Exercise 15

The battery in Exercise 13 is replaced by a generator producing a voltage of $E(t) = 12\sin 10t$. Find the charge at time t.

Solution

The equation for the charge in a circuit consisting of an inductor, a resistor, and a capacitor in series with this generator is given by

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = 12\sin 10t.$$

Since there's zero charge and no current initially, the initial conditions associated with this ODE are $Q(0) = 0$ and $Q'(0) = 0$. Because the ODE is linear, the general solution can be written as the sum of a complementary solution and a particular solution.

$$Q = Q_c + Q_p$$

The complementary solution satisfies the associated homogeneous equation.

$$L\frac{d^2Q_c}{dt^2} + R\frac{dQ_c}{dt} + \frac{1}{C}Q_c = 0. \tag{1}$$

Because this ODE is homogeneous and has constant coefficients, it has solutions of the form $Q_c = e^{rt}$.

$$Q_c = e^{rt} \quad \rightarrow \quad \frac{dQ_c}{dt} = re^{rt} \quad \rightarrow \quad \frac{d^2Q_c}{dt^2} = r^2e^{rt}$$

Substitute these formulas into equation (1).

$$L(r^2e^{rt}) + R(re^{rt}) + \frac{1}{C}(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$Lr^2 + Rr + \frac{1}{C} = 0$$

Multiply both sides by C.

$$LCr^2 + RCr + 1 = 0$$

Solve for r, noting that $R^2C^2 - 4LC < 0$.

$$r = \frac{-RC \pm i\sqrt{4LC - R^2C^2}}{2LC}$$

$$r = \left\{\frac{-RC - i\sqrt{4LC - R^2C^2}}{2LC}, \frac{-RC + i\sqrt{4LC - R^2C^2}}{2LC}\right\}$$

Two solutions to the ODE are

$$\exp\left(\frac{-RC - i\sqrt{4LC - R^2C^2}}{2LC}t\right) \quad \text{and} \quad \exp\left(\frac{-RC + i\sqrt{4LC - R^2C^2}}{2LC}t\right).$$

According to the principle of superposition, the general solution to equation (1) is a linear combination of these two.

$$\begin{aligned}
Q_c(t) &= C_1 \exp\left(\frac{-RC - i\sqrt{4LC - R^2C^2}}{2LC}t\right) + C_2 \exp\left(\frac{-RC + i\sqrt{4LC - R^2C^2}}{2LC}t\right) \\
&= C_1 \exp\left(-\frac{R}{2L}t\right) \exp\left(-i\frac{\sqrt{4LC - R^2C^2}}{2LC}t\right) + C_2 \exp\left(-\frac{R}{2L}t\right) \exp\left(i\frac{\sqrt{4LC - R^2C^2}}{2LC}t\right) \\
&= \exp\left(-\frac{R}{2L}t\right)\left[C_1 \exp\left(-i\frac{\sqrt{4LC - R^2C^2}}{2LC}t\right) + C_2 \exp\left(i\frac{\sqrt{4LC - R^2C^2}}{2LC}t\right)\right] \\
&= \exp\left(-\frac{R}{2L}t\right)\left[C_1\left(\cos\frac{\sqrt{4LC - R^2C^2}}{2LC}t - i\sin\frac{\sqrt{4LC - R^2C^2}}{2LC}t\right)\right. \\
&\qquad \left. + C_2\left(\cos\frac{\sqrt{4LC - R^2C^2}}{2LC}t + i\sin\frac{\sqrt{4LC - R^2C^2}}{2LC}t\right)\right] \\
&= \exp\left(-\frac{R}{2L}t\right)\left[(C_1 + C_2)\cos\frac{\sqrt{4LC - R^2C^2}}{2LC}t + (-iC_1 + iC_2)\sin\frac{\sqrt{4LC - R^2C^2}}{2LC}t\right] \\
&= \exp\left(-\frac{R}{2L}t\right)\left(C_3\cos\frac{\sqrt{4LC - R^2C^2}}{2LC}t + C_4\sin\frac{\sqrt{4LC - R^2C^2}}{2LC}t\right)
\end{aligned}$$

On the other hand, the particular solution satisfies the original ODE.

$$L\frac{d^2Q_p}{dt^2} + R\frac{dQ_p}{dt} + \frac{1}{C}Q_p = 12\sin 10t \tag{2}$$

The inhomogeneous term is a sine function, so the trial solution is $Q_p = A\cos 10t + B\sin 10t$.

$$Q_p = A\cos 10t + B\sin 10t \quad \rightarrow \quad \frac{dQ_p}{dt} = -10A\sin 10t + 10B\cos 10t \quad \rightarrow \quad \frac{d^2Q_p}{dt^2} = -100A\cos 10t - 100B\sin 10t$$

Substitute these formulas into equation (2).

$$L(-100A\cos 10t - 100B\sin 10t) + R(-10A\sin 10t + 10B\cos 10t) + \frac{1}{C}(A\cos 10t + B\sin 10t) = 12\sin 10t$$

$$\frac{A - 100ALC + 10BRC}{C}\cos 10t + \frac{B - 100BLC - 10ARC}{C}\sin 10t = 12\sin 10t$$

Match the coefficients on both sides to get a system of equations for A and B.

$$\frac{A - 100ALC + 10BRC}{C} = 0$$

$$\frac{B - 100BLC - 10ARC}{C} = 12$$

Solving it yields

$$A = -\frac{120RC^2}{(1 - 100LC)^2 + 100R^2C^2} \quad \text{and} \quad B = \frac{12C(1 - 100LC)}{(1 - 100LC)^2 + 100R^2C^2}.$$

The particular solution is then

$$Q_p = -\frac{120RC^2}{(1-100LC)^2 + 100R^2C^2}\cos 10t + \frac{12C(1-100LC)}{(1-100LC)^2 + 100R^2C^2}\sin 10t,$$

and the general solution to the original ODE is

$$\begin{aligned}
Q(t) &= Q_c + Q_p \\
&= \exp\left(-\frac{R}{2L}t\right)\left(C_3\cos\frac{\sqrt{4LC - R^2C^2}}{2LC}t + C_4\sin\frac{\sqrt{4LC - R^2C^2}}{2LC}t\right) \\
&\qquad - \frac{120RC^2}{(1-100LC)^2 + 100R^2C^2}\cos 10t + \frac{12C(1-100LC)}{(1-100LC)^2 + 100R^2C^2}\sin 10t.
\end{aligned}$$

Differentiate it with respect to t.

$$\begin{aligned}
\frac{dQ}{dt} &= -\frac{R}{2L}\exp\left(-\frac{R}{2L}t\right)\left(C_3\cos\frac{\sqrt{4LC - R^2C^2}}{2LC}t + C_4\sin\frac{\sqrt{4LC - R^2C^2}}{2LC}t\right) \\
&\qquad + \exp\left(-\frac{R}{2L}t\right)\left(-C_3\frac{\sqrt{4LC - R^2C^2}}{2LC}\sin\frac{\sqrt{4LC - R^2C^2}}{2LC}t\right. \\
&\qquad\qquad \left. +C_4\frac{\sqrt{4LC - R^2C^2}}{2LC}\cos\frac{\sqrt{4LC - R^2C^2}}{2LC}t\right) \\
&\qquad + \frac{1200RC^2}{(1-100LC)^2 + 100R^2C^2}\sin 10t + \frac{120C(1-100LC)}{(1-100LC)^2 + 100R^2C^2}\cos 10t
\end{aligned}$$

Apply the initial conditions to determine C_3 and C_4.

$$\begin{aligned}
Q(0) &= C_3 - \frac{120RC^2}{(1-100LC)^2 + 100R^2C^2} = 0 \\
\frac{dQ}{dt}(0) &= -\frac{R}{2L}C_3 + C_4\frac{\sqrt{4LC - R^2C^2}}{2LC} + \frac{120C(1-100LC)}{(1-100LC)^2 + 100R^2C^2} = 0
\end{aligned}$$

Solving this system yields

$$C_3 = \frac{120RC^2}{(1-100LC)^2 + 100R^2C^2} \quad \text{and} \quad C_4 = \frac{120C^2(-2L + 200CL^2 + CR^2)}{\sqrt{4LC - R^2C^2}[(1-100LC)^2 + 100R^2C^2]}.$$

Therefore,

$$Q(t) = \exp\left(-\frac{R}{2L}t\right)\left[\frac{120RC^2}{(1-100LC)^2 + 100R^2C^2}\cos\frac{\sqrt{4LC - R^2C^2}}{2LC}t\right.$$
$$\left. + \frac{120C^2(-2L + 200CL^2 + CR^2)}{\sqrt{4LC - R^2C^2}[(1-100LC)^2 + 100R^2C^2]}\sin\frac{\sqrt{4LC - R^2C^2}}{2LC}t\right]$$
$$- \frac{120RC^2}{(1-100LC)^2 + 100R^2C^2}\cos 10t + \frac{12C(1-100LC)}{(1-100LC)^2 + 100R^2C^2}\sin 10t.$$

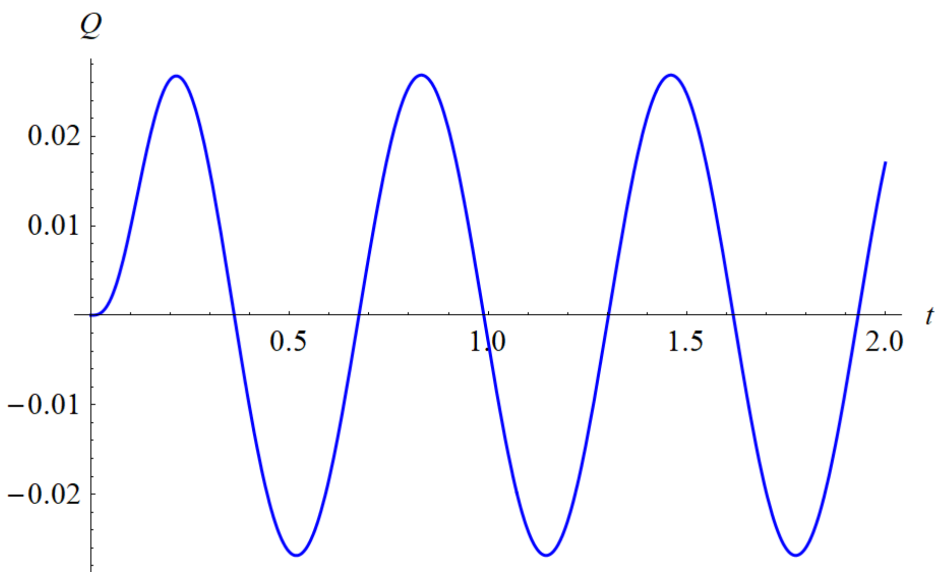
Plug in $R = 20\ \Omega$, $L = 1$ H, and $C = 0.002$ F.

$$Q(t) = e^{-10t}(0.012\cos 20t - 0.006\sin 20t) - 0.012\cos 10t + 0.024\sin 10t$$

Differentiate this with respect to t to get the current.

$$\begin{aligned} I(t) = \frac{dQ}{dt} &= -10e^{-10t}(0.012\cos 20t - 0.006\sin 20t) + e^{-10t}(-0.24\sin 20t - 0.12\cos 20t) + 0.12\sin 10t + 0.24\cos 10 \\ &= e^{-10t}(-0.24\cos 20t - 0.18\sin 20t) + 0.24\cos 10t + 0.12\sin 10t \end{aligned}$$

Below is a plot of the charge versus time.



Below is a plot of the current versus time.